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Conditional Bounds and Best L_∞-Approximations in Probability Spaces

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The paper deals with L_∞-approximations in a probability space (Ω, σ, P) by means of α-measurable random variables for $\alpha \subset \sigma$, a σ-lattice. Attention is paid to the characterization of the set of all best L_∞-approximations in terms of the notion of "conditional bounds," developed in the paper. On the other hand we study in the framework above the Pólya algorithm, showing that, if f_r denotes a best L_r-approximation and $r(n) \to \infty$, then $\liminf f_{r(n)}$ and $\limsup f_{r(n)}$ are best L_∞-approximations. We also point out an error in an article on this subject by Darst and discuss the validity of subsequent articles by Darst, Al-Rashed, and others. © 1989 Academic Press, Inc.

1. INTRODUCTION

This paper deals with two distinct aspects of L_∞-approximations. The first is the study and characterization of the set of best L_∞-approximations to a random variable by elements of $L_\infty(\Omega, \alpha, P)$, where α is a σ-lattice. The second is the practical attainment of these best L_∞-approximations. In this direction we provide a complete characterization in the case of simple random variables and discuss the validity of the Pólya algorithm.

In [9], Darst studies the convergence, as $r \to \infty$, of the conditional r-means *given a σ-algebra*. The limit of the best L_r-approximations to X by α-measurable functions, as $r \to \infty$, is called by Darst the best best L_∞-approximation to X by elements of $B = L_\infty(\Omega, \alpha, P)$, in the sense that for each $E \in \alpha$ the restriction of this element to E is a best L_∞-approximation

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to the restriction of X to E. The technique employed by Darst is based on the use of partitions of the space by means of suitable sets. However, the proof of the main result in [9] is not correct. More precisely, Theorem 1 in Darst's paper is a direct consequence of his Lemma 4 whose derivation in [9] is not completely satisfactory. Darst's work has been followed in [1, 2, 11].

In Section 2 we present some notations and definitions and discuss the difficulties that arise from the incorrectness of Lemma 4 in [9].

In Section 3 we present an alternative technique for proving results similar to those in [1, 2]. We would like to emphasize that the use of conditional bounds, introduced in this paper, provides stronger results than those of [1, 2] with rather simple proofs (we work in the general framework of the L_∞-approximation by measurable functions given a σ-lattice).

For example, the following statement appears in [1]: "Let $\{\alpha_n\}$ be an increasing sequence of sub-σ-algebras of σ, and let α_∞ be the σ-algebra generated by $\bigcup_n \alpha_n$. Call $B_n = L_\infty(\Omega, \alpha_n, P)$, $n = 1, ..., \infty$; and let f_n, $n = 1, ..., \infty$ be the best best L_∞-approximation to X by elements of B_n. Then $\{f_n\}$ converges a.e. but not necessarily to f_∞." With our technique this result becomes trivial and a characterization of the limit is possible. We also prove the convergence in the case where the sequence of sub-σ-algebras is decreasing. The comparison between the proof in [2] and the proof we present in Corollary 3.8 clearly shows the advantages of our technique.

Theorem 3.3 proves that the set of all best L_∞-approximations to X by elements of the closed convex cone $B \equiv L_\infty(\Omega, \alpha, P)$, $\mathscr{A}_\infty$, is not empty. This theorem also shows that $\mathscr{A}_\infty$ is an interval of B and characterizes $\mathscr{A}_\infty$ in terms of the conditional bounds given α, L_α, and U_α.

Explicit expressions for $d = d(X, B)$ and $\Phi = f^* - f_*$ (the difference between the extremes of the interval $\mathscr{A}_\infty = \{g \in B; f_* \leqslant g \leqslant f^*\}$) are obtained in Theorem 3.4 and Corollary 3.5.

Theorem 3.6 proves that, when α is a σ-algebra, most results in this paper can be restated in terms of regular conditional distributions.

We remark that when α is a σ-lattice the study of the conditional midrange (the best best L_∞-approximation when α is a σ-algebra) appears to be difficult via techniques based on partitions, such as those employed in [9].

Section 4 is devoted to the second aspect mentioned at the beginning of this introduction: The Pólya algorithm attempts to obtain a best L_∞-approximation as the limit, as $r \to \infty$, of the best L_r-approximations. In [7], we have proved that, if α is a σ-algebra, then the best L_r-approximation of a function X by α-measurable functions (or conditional r-means given α) converge a.s. to the conditional midrange. Moreover the latter is a best L_∞-approximation by α-measurable functions, thus proving that in that case the Pólya algorithm is successful. On the other hand Darst *et al.* [10, 11] have shown, by means of examples, that the Pólya algorithm may fail in the case of monotone approximations: the best L_r-approximations by monotone functions are not necessarily convergent as $r \to \infty$. The question remains whether the limit of a *convergent sequence* of best L_r-approximations is a best L_∞-approximation. More generally, if $r_n \to \infty$ and $f_{r(n)}$ is the best $L_{r(n)}$-approximation, are the functions $\liminf f_{r(n)}$, $\limsup f_{r(n)}$ best L_∞-approximations? On the other hand, in [7] we have shown that $\lim_{r \to \infty} \|X - f_r\|_r = \|X - f_\infty\|_\infty$ when α is a σ-algebra. Does this result remain valid when α is only a σ-lattice?

In this paper we answer in the affirmative the previous questions. We work on probability spaces, in the general framework of conditional r-means given a σ-lattice, which originates from Brunk [3, 4] and includes as particular cases those conditional r-means given a σ-algebra and the isotonic and monotone approximations. Note that our results imply that if the best L_∞-approximation is unique then it can be computed by the Pólya algorithm.

2. Notation and Preliminary Results

In this paper (Ω, σ, P) denotes a probability space and α a sub-σ-lattice of σ (α is closed under countable unions and intersections and $\Omega \in \alpha$, $\varnothing \in \alpha$). An extended real function $f: \Omega \to R^*$ is α-measurable if $\{f > a\} \in \alpha$ for all $a \in R$. By $L_r(\alpha) \equiv L_r(\Omega, \alpha, P)$, $1 \leqslant r \leqslant \infty$, we denote the system of all equivalence classes in $L_r(\Omega, \sigma, P)$ containing an α-measurable function. Often we will not distinguish between a function and the equivalence class it represents.

Let X be a random variable belonging to $L_\infty(\sigma)$. The L_∞-distance from X to the closed convex cone $B = L_\infty(\alpha)$ will be denoted by $d = d(X, B) = \inf\{\|X - g\|_\infty;\ g \in B\}$. Denote by $\mathscr{A}_\infty \equiv \mathscr{A}_\infty(X, B)$ the set of all best L_∞-approximations to X by elements of B.

The conditional r-mean given the σ-lattice α, $1 < r < \infty$, is the (unique) best L_r-approximation to X by elements of $L_r(\alpha)$ (see [13] for a complete study).

A well-known result, which will be used later, is the following (see, e.g., [6, p. 190]):

(I) *If* $\{f_i,\ i \in I\}$ *is a family of random variables on* (Ω, σ, P), $f^* = \operatorname{essup}_{i \in I} f_i$ (*resp.* $f_* = \operatorname{esinf}_{i \in I} f_i$) *denotes the random variable defined, up to P-equivalence, by the relation*: *If g is* σ-*measurable, then*

$$f_i \leqslant g \text{ (resp. } f_i \geqslant g) \text{ a.s. } i \in I \iff f^* \leqslant g \text{ (resp. } f_* \geqslant g) \text{ a.s.}$$

It can be proved that there exist countable sets $M \subset I$, $N \subset I$ such that $f^* = \sup_{i \in M} f_i$ and $f_* = \inf_{i \in N} f_i$.

If $\{f_n\}$ is a sequence of α-measurable functions, then $\sup f_n$ and $\inf f_n$ are α-measurable. Therefore, if $\{f_i, i \in I\}$ is a family of α-measurable functions, $\mathrm{essup}_{i \in I} f_i$ and $\mathrm{esinf}_{i \in I} f_i$ are also α-measurable functions.

If $\{C_i, i \in I\}$ is a family of sets in σ, the sets $C^* = \mathrm{essup}_{i \in I} C_i$ and $C_* = \mathrm{esinf}_{i \in I} C_i$ are defined by the relations:

$$C^* = \{\mathrm{essup}_{i \in I} I_{C_i} = 1\}, \qquad C_* = \{\mathrm{esinf}_{i \in I} I_{C_i} = 1\}$$

(I_A denotes the indicator function of the set A).

We define the α-conditional essential infimum (ei) of X as the α-measurable function $L_\alpha = \mathrm{essup}\{g;\, g$ is α-measurable and $g \leqslant X$ a.e.$\}$. In a similar way the α-conditional essential supremum (es) of X is defined by $U_\alpha = \mathrm{esinf}\{g;\, g$ is α-measurable and $g \geqslant X$ a.e.$\}$. The α-conditional midrange of X is defined by $M_\alpha = \frac{1}{2}(L_\alpha + U_\alpha)$.

As we have announced before, the proof of the main result in [9] is not correct. Darst has kindly pointed out to us that it is possible to modify the proof of his Theorem 1 to obtain only L_1-convergence from a new statement of Lemma 4. Darst's new version for Lemma 4 is:

> "*Suppose $\mu(F_i) > 0$, $\delta_i > 0$, $\varepsilon_i > 0$. Then there exists $p_i = p_i(F_i, \delta_i, \varepsilon_i)$ such that $p \geqslant p_i$ implies $\mu\{x \in F_i;\, |f_p(x) - m_i| > \gamma/2 + \varepsilon_i\} < \delta_i$, where $m_i = \frac{1}{2}\{\mathrm{esinf}(f, F_i) + \mathrm{essup}(f, F_i)\}$.*"

Similar corrections can also be made in [11] to obtain L_1-convergence. We have proved the a.e. convergence in [7]. On the contrary, the results in [1, 2] are essentially rightly obtained because the modification of Lemma 4 and the L_1-convergence suffice for the proofs there.

The difficulties for proving a.e. convergence in Theorem 1 of [9] can be circumvented as follows (cf. [7]);

(II) THEOREM. *Let $P_x(A, \omega) \equiv P_x^\alpha(A, \omega)$ be a regular conditional distribution for X given the σ-algebra α (see, for example,* [6, p. 213]). *Define the conditional midrange, M_α, of X given α by means of $M_\alpha(\omega) = \frac{1}{2}(\omega\text{-essup} + \omega\text{-esinf})$ where* ω-essup (*resp.* ω-esinf) *is the essential supremum (resp. infimum) value of the identity in R for the $P_x(\cdot, \omega)$-probability.*

Then there exist versions g_r of the conditional r-means of X given α such that $g_r \to M_\alpha$ a.e. as $r \to \infty$. Moreover, M_α is a best L_∞-approximation to X by elements of B.

Proof. Observe that we can suppose w.l.o.g. that the identity in R is $P_x(\cdot, \omega)$-a.s. bounded for P-a.e. $\omega \in \Omega$. Let a_-^s denote $|a|^s \cdot \mathrm{sign}(a)$ and let h_r

be the conditional r-mean of X given α. Recall that h_r is characterized by (see [13])

$$h_r \in L_r(\Omega, \sigma, P) \qquad \text{and} \qquad \int g(X-h_r)^{\underline{r-1}}\, dP = 0 \qquad \text{for any } g \in L_r(\Omega, \alpha, P).$$

A standard reasoning (begin with $H = I_{B\times A}$, B a Borel set and $A \in \alpha$) shows that if $H: R\times\Omega \to R$ is (Borel xα)-measurable and $H(X, id) \in L_1(\Omega, \sigma, P)$ then $\psi(\omega) = \int H(t, \omega)\, P_x(dt, \omega)$ is a version of the conditional mean of $H(X(\cdot), \cdot)$ given α. Putting $H(t, \omega) = (t - h_r(\omega))^{\underline{r-1}}$, we see that the map $\psi_r: \Omega \to R$, $\psi_r(\omega) = \int (t - h_r(\omega))^{\underline{r-1}} P_x(dt, \omega)$ is the conditional mean (or 2-mean) of $(X - h_r)^{\underline{r-1}}$ given α; hence by the characterization above of the conditional r-mean: $\psi_r = 0$ P-a.e.

Let $g_r(\omega)$ be the value of the r-mean of the identity on R with respect to the probability $P_x(\cdot, \omega)$. Then

$$\int (t - g_r(\omega))^{\underline{r-1}} P_x(dt, \omega) = 0 \qquad \text{for every} \quad \omega \in \Omega,$$

so $h_r = g_r P$-a.e.

By the convergence of the r-mean to the midrange as $r \to \infty$ (see [8]) we have then: $\lim_{r\to\infty} g_r(\omega) = \frac{1}{2}(\omega\text{-esinf} + \omega\text{-essup}) = M_\alpha(\omega)$.

A proof of the fact that M_α is a best L_∞-approximation is given in [7], but a simpler one will be given in Theorem 3.3 taking into account Theorem 3.6.

3. Best L_∞-Approximations and Conditional Bounds

Some obvious properties of the α-conditionals es and ei are stated in the following proposition:

PROPOSITION 3.1. (a) *$L_\alpha \leqslant X \leqslant U_\alpha$ a.e. and if g is an α-measurable function verifying $g \leqslant X$ a.e. (resp. $g \geqslant X$ a.e.) then $g \leqslant L_\alpha$ a.e. (resp. $g \geqslant U_\alpha$). Moreover, for each $k \in R$, $L_\alpha + k$ (resp. $U_\alpha + k$) is the α-conditional ei (resp. es) of $X + k$.*

(b) *L_α, U_α, M_α belong to $L_\infty(\Omega, \alpha, P)$.*

(c) *Let α, β be two sub-σ-lattices of σ and suppose $\alpha \subset \beta$; then $L_\alpha \leqslant L_\beta$ and $U_\alpha \geqslant U_\beta$ a.e.*

If, moreover, α is a σ-algebra:

(d) *L_α (resp. U_α) is the unique (up to equivalences) α-measurable function with $P(L_\alpha \leqslant X/\alpha) = 1$ a.e. (resp. $P(U_\alpha \geqslant X/\alpha) = 1$ a.e.) that verifies* *the following property: If f is α-measurable and $P(f \leqslant X/\alpha)=1$ a.e. (resp. $P(f \geqslant X/\alpha)=1$ a.e.) then $L_\alpha \geqslant f$ a.e. (resp. $U_\alpha \leqslant f$ a.e.).*

When α is a σ-algebra, $\mathscr{A}_\infty(X, B)$ is not empty. This is proved in [7] by use of the existence of the limit of the conditional r-mean as $r \to \infty$. This limit may not exist if α is a σ-lattice but not a σ-algebra (see [11]). Therefore we prove the existence of elements of best L_∞-approximation to X in B in a different way.

We first need a lemma. Let $m>0$ and call $\mathscr{C}_m=\{g;\ g \in B$ and $\|X-g\|_\infty \leqslant m\}$.

LEMMA 3.2. *Assume $\mathscr{C}_m$ not empty. Then $\mathscr{C}_m$ coincides with the set $\{g;\ g$ is α-measurable and $U_\alpha - m \leqslant g \leqslant L_\alpha + m$ a.e.$\}$.*

Proof. Note first that $\mathscr{C}_m = \{g;\ g \in B,\ X-m \leqslant g \leqslant X+m$ a.e.$\}$. Now, on taking $f_{*m} = \operatorname{esinf}\{g;\ g \in \mathscr{C}_m\}$, $f^*_m = \operatorname{essup}\{g;\ g \in \mathscr{C}_m\}$; it suffices to prove $f_{*m} = U_\alpha - m$ and $f^*_m = L_\alpha + m$ a.e.

We only prove $f_{*m} = U_\alpha - m$ a.e., as the proof of $f^*_m = L_\alpha + m$ is similar.

From Proposition 3.1(a), the α-conditional es of the random variable $X-m$ is $U^m_\alpha = U_\alpha - m$. Moreover Proposition 3.1(a) implies that $U^m_\alpha \leqslant g$ a.e. for each $g \in \mathscr{C}_m$, whence $U^m_\alpha \leqslant \operatorname{esinf}\{g;\ g \in \mathscr{C}_m\} = f_{*m}$ a.e.

Now observe that U^m_α is α-measurable and, as $f_{*m} \leqslant g$ a.e. for each $g \in \mathscr{C}_m$ (not empty), we have $f_{*m} \leqslant X+m$ a.e. Hence $X-m \leqslant U^m_\alpha(\leqslant f_{*m}) \leqslant X+m$ a.e., whence $f_{*m} \leqslant U^m_\alpha$ is α-measurable.

The following theorem characterizes the set $\mathscr{A}_\infty(X, B)$.

THEOREM 3.3. *$\mathscr{A}_\infty = \mathscr{A}_\infty(X, B)$ is not empty (in fact the conditional midrange belongs to $\mathscr{A}_\infty$). Moreover $\mathscr{A}_\infty$ coincides with the set $\{g;\ g$ α-measurable and $U_\alpha - d \leqslant g \leqslant L_\alpha + d$ a.e.$\}$.*

Proof. It is obvious that $\mathscr{A}_\infty = \mathscr{C}_d = \bigcap_{n \in N} \mathscr{C}_{d+(1/n)}$ and that $\mathscr{C}_{d+(1/n)}$ is not empty for every $n \in N$. Lemma 3.2 yields

$$\mathscr{C}_{d+(1/n)} = \{g;\ g\ \alpha\text{-measurable and } U_\alpha - d - 1/n \leqslant g \leqslant L_\alpha + d + 1/n \text{ a.e.}\},$$

whence $M_\alpha = \frac{1}{2}(L_\alpha + U_\alpha) = \frac{1}{2}\{(L_\alpha + d + 1/n) + (U_\alpha - d - 1/n)\} \in \mathscr{C}_{d+(1/n)}$ for every $n \in N$. Thus $M_\alpha \in \mathscr{A}_\infty$ and $\mathscr{A}_\infty$ is not empty.

The application of Lemma 3.2 to $\mathscr{C}_d$ finishes the proof.

The distance, d, of the variable X to the set B may also be characterized by means of L_α and U_α:

THEOREM 3.4. $d = d(X, B) = \frac{1}{2}\, \|U_\alpha - L_\alpha\|_\infty$.

Proof. Set $\imath = \frac{1}{2}\|U_\alpha - L_\alpha\|_\infty$. Theorem 3.3 implies that $U_\alpha - d \leqslant L_\alpha + d$ a.e., hence $0 \leqslant U_\alpha - L_\alpha \leqslant 2d$ a.e. and $2\imath = \|U_\alpha - L_\alpha\|_\infty \leqslant 2d$.

We shall prove that $\imath \geqslant d$. Observe that:

$$\|U_\alpha - \tfrac{1}{2}(U_\alpha + L_\alpha)\|_\infty = \|\tfrac{1}{2}(U_\alpha + L_\alpha) - L_\alpha\|_\infty = \|\tfrac{1}{2}(U_\alpha - L_\alpha)\|_\infty = \imath.$$

In Theorem 3.3 we have also obtained that $\frac{1}{2}(U_\alpha + L_\alpha) \in \mathscr{A}_\infty$, which implies $d = \|X - \frac{1}{2}(U_\alpha + L_\alpha)\|_\infty$.

Moreover:

(a) If $X \geqslant \frac{1}{2}(U_\alpha + L_\alpha)$, then $|X - \frac{1}{2}(U_\alpha + L_\alpha)| = X - \frac{1}{2}(U_\alpha + L_\alpha) \leqslant U_\alpha - \frac{1}{2}(U_\alpha + L_\alpha) \leqslant \|U_\alpha - \frac{1}{2}(U_\alpha + L_\alpha)\|_\infty = \imath$.

(b) If $X < \frac{1}{2}(U_\alpha + L_\alpha)$, then $|X - \frac{1}{2}(U_\alpha + L_\alpha)| = \frac{1}{2}(U_\alpha + L_\alpha) - X \leqslant \frac{1}{2}(U_\alpha + L_\alpha) - L_\alpha \leqslant \|\frac{1}{2}(U_\alpha + L_\alpha) - L_\alpha\|_\infty = \imath$.

Thus $d = \|X - \frac{1}{2}(U_\alpha + L_\alpha)\|_\infty \leqslant \imath$.

Let f_* (resp. f^*) be the smallest (resp. the largest) best L_∞-approximation to X by elements of B (then $f_* = U_\alpha - d$ and $f^* = L_\alpha + d$). Reference [1] provides, in the case where α is a σ-algebra, a characterization of the difference $\Phi = f^* - f_*$ in terms of the sets developed in [9]. We present an explicit characterization of Φ in the following obvious corollary:

COROLLARY 3.5. *Let $\Phi = f^* - f_*$ be the difference between the extremal best L_∞-approximations to X by elements of B. Then:*

$$\Phi = \|U_\alpha - L_\alpha\|_\infty - (U_\alpha - L_\alpha) = 2d - (U_\alpha - L_\alpha).$$

THEOREM 3.6. *Let α be a sub-σ-algebra of σ, let $P_x(A, \omega) \equiv P_x^\alpha(A, \omega)$ be a regular conditional distribution for X given α and let $F(t, \omega)$ be the associated conditional distribution function. Define L^* by $L^*(\omega) = \inf\{t/F(t, \omega) > 0\}$ (resp. U^* by $U^*(\omega) = \sup\{t/F(t, \omega) < 1\}$). Then L^* (resp. U^*) is a version of the α-conditional ei (resp. es) of X.*

Proof. We only prove the statement for L^*; the other case is similar. Let $\{t_n\}$ be the set, Q, of rational numbers. Define Γ_n by

$$\Gamma_n(\omega) = \begin{cases} t_n, & \text{if } F(t_n, \omega) = 0 \\ -\infty, & \text{if } F(t_n, \omega) > 0. \end{cases}$$

Obviously Γ_n is α-measurable and $L^* = \sup \Gamma_n$, and therefore L^* is α-measurable.

For every n:

$$P\{X \geqslant \Gamma_n/\alpha\} \overset{\text{a.e.}}{=} = P\{X \geqslant t_n, F(t_n, \cdot) = 0/\alpha\} + P\{X \geqslant -\infty, F(t_n, \cdot) > 0/\alpha\}$$
$$\overset{\text{a.e.}}{=} = I_{\{F(t_n, \cdot) = 0\}}(1 - F(t_n, \cdot)) + I_{\{F(t_n, \cdot\}} = 1.$$

Thus $P\{X \geqslant L^*/\alpha\} = 1$ a.e. Now, taking into account Proposition 3.1(d), it suffices to show that for f α-measurable with $P\{X \geqslant f/\alpha\} = 1$ a.e. one has $L^* \geqslant f$ a.e.

Assume, on the contrary, $P(A) > 0$ where $A = \{L^* < f\}$. Since $A = \bigcup_{S \in Q} \{L^* < s < f\}$ then there exists a rational number, s, such that $P\{L^* < s < f\} > 0$. But $L^*(\omega) < s$ implies $F(s, \omega) > 0$, hence:

$$P\{X < f/\alpha\} \overset{\text{a.e.}}{\geqslant} \geqslant P\{X < s < f/\alpha\} \overset{\text{a.e.}}{=} = I_{(s<f)} P\{X < s/\alpha\} \overset{\text{a.e.}}{=}$$
$$= I_{(s<f)} F(s, \cdot) \geqslant I_{\{L^* < s < f\}} F(s, \cdot).$$

Thus we have $P\{P(X < f/\alpha) > 0\} \geqslant P\{L^* < s < f\} > 0$.

Therefore $P\{P(X \geqslant f/\alpha) < 1\} \neq 0$ contradicting that $P\{X \geqslant f/\alpha\} = 1$ a.e.

It is obvious from this theorem that, in the case where α is a σ-algebra, the conditional midrange, M_α, of X given α, as defined in (II), is a version of the α-conditional midrange. Therefore (II) shows that the α-conditional midrange coincides with the best best L_∞-approximation by elements of $L_\infty(\Omega, \alpha, P)$ (α being a σ-algebra).

Now we present some convergence results.

THEOREM 3.7. *Let $\{\alpha_n\}$ be an increasing (resp. decreasing) sequence of sub-σ-lattices of σ and let α_∞ be the σ-lattice generated by $\bigcup_n \alpha_n$ (resp. $\alpha_\infty = \bigcap_n \alpha_n$). Let L_n and U_n be the α_n-conditionals ei and es for the random variable X. Then, $L_n \uparrow \sup L_n \leqslant L_\infty$ a.e. and $U_n \downarrow \inf U_n \geqslant U_\infty$ a.e. (resp. $L_n \downarrow L_\infty$ a.e. and $U_n \uparrow U_\infty$ a.e.), where L_∞ and U_∞ are the α_∞-conditionals ei and es for X.*

Proof. The increasing case follows from Proposition 3.1(c). The example in [2] proves that U_n need not converge to U_∞. Examples in which L_n does not converge to L_∞ can be obtained by similar methods.

Now suppose $\alpha_n \downarrow \alpha_\infty$. We prove $L_n \downarrow L_\infty$ a.e., because the proof of $U_n \uparrow U_\infty$ is analogous.

From Proposition 3.1(c) we obtain $L_n \downarrow \inf L_n$ a.e. and $\inf L_n \geqslant L_\infty$ a.e. Obviously $\inf L_n$ is α_k-measurable for every k, and so $\inf L_n$ is α_∞-measurable. Also $P\{X \geqslant \inf L_n\} \geqslant P\{X \geqslant L_k\} = 1$ for every k, so that $\inf L_n \leqslant L_\infty$ a.e. Therefore $\inf L_n = L_\infty$ a.e.

COROLLARY 3.8. *With the notation and hypotheses of the previous theorem, let* M_n, $n=1, 2, ..., \infty$, *be the* α_n*-conditional midrange of* X. *Then* M_n *converges a.e. Moreover, in the decreasing case* $(\alpha_n \downarrow \alpha_\infty)$ *we have* $M_n \to M_\infty$ *a.e.*

In the particular case where it is assumed that α_n, $n=1, 2, ..., \infty$, are σ-algebras, this corollary proves the a.e. convergence of the best best L_∞-approximants to X by elements of $L_\infty(\Omega, \alpha_n, P)$.

THEOREM 3.9. *Let* $\{X_n\}$ *be a sequence of random variables in* $L_\infty(\Omega, \sigma, P)$ *such that* $X_n \to X$ *in the* L_∞*-norm, and let* α *be a sub-*σ*-lattice of* σ. *Now denote by* L_n, U_n, M_n *(resp.* L, U, M*) the* α*-coditionals ei, es and the midrange for* X_n *(resp.* X*). Then* $L_n \to L$, $U_n \to U$, *and* $M_n \to M$ *in the* L_∞*-norm. (If* α *is a* σ*-algebra the best* L_∞*-approximation by elements of* $L_\infty(\Omega, \alpha, P)$ *is continuous in* $L_\infty(\Omega, \sigma, P)$.)

Proof. We present the proof for the sequence $\{L_n\}$ only.

Take $\delta > 0$. The convergence $X_n \to X$ in the L_∞-norm implies that there exists n_0 such that $|X_n - X| \leqslant \delta$ a.e. for every $n \geqslant n_0$. Therefore, if $n \geqslant n_0$ we can write

$$P\{X_n \geqslant L - \delta\} \geqslant P\{X_n \geqslant X - \delta \geqslant L - \delta\} = 1,$$

and

$$P\{X \geqslant L_n - \delta\} \geqslant P\{X \geqslant X_n - \delta \geqslant L_n - \delta\} = 1.$$

Hence $L_n - \delta \leqslant L$ a.e. and $L - \delta \leqslant L_n$ a.e., and so $|L_n - L| \leqslant \delta$ a.e.

4. THE PÓLYA ALGORITHM IN ISOTONIC REGRESSION

The main result on this topic is contained in Theorem 4.4. Proposition 4.1 characterizes in a simple way the set $\mathscr{A}_\infty(X, B)$ for a simple random variable. Finally we prove in Theorem 4.5 that the L_r-distance from X to $L_r(\alpha)$ converges to $d = d(X, B)$, the L_∞-distance between X and B. Recall that X is a P-a.s. bounded random variable and assume that, from now on, L (resp. U) denotes the α-conditional ei (resp. es) of X and f_r is the α-conditional r-mean of X given α.

PROPOSITION 4.1. *Let* $X = \sum_{i=1}^n \lambda_i I_{A_i}$ *be a simple random variable where the* A_i *are disjoint sets whose union is* Ω $(A_1 + \cdots + A_n = \Omega)$, *and suppose* $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. *Define the sets* ${}^*C_k = \text{esinf}\{C \in \alpha; A_k + A_{k+1} + \cdots + A_n \subset C\}$, ${}_*C_k = \text{essup}\{C \in \alpha;\ C \subset A_k + A_{k+1} + \cdots + A_n\}$. *Then* $L = \sum_{i=1}^n \lambda_i I_{{}_*C_i - {}_*C_{i+1}}$ *and* $U = \sum_{i=1}^n \lambda_i I_{{}^*C_i - {}^*C_{i+1}}$ *(where* ${}^*C_{n+1} = {}_*C_{n+1} = \varnothing$*).*

Proof. Let us denote $L^* = \sum_{i=1}^{n} \lambda_i I_{{}_*C_i - {}_*C_{i+1}}$ and $U^* = \sum_{i=1}^{n} \lambda_i I_{{}^*C_i - {}^*C_{i+1}}$. First we prove $U = U^*$. Obviously U^* is α-measurable, and, being $\sum_{i \geqslant 1}({}^*C_i - {}^*C_{i+1}) = \Omega$, it suffices to show that $g \geqslant \lambda_i$ a.e. on ${}^*C_i - {}^*C_{i+1}$ for each α-measurable function g verifying $g \geqslant X$ a.e. Let g be such a function. Since $\lambda_i < \lambda_{i+1} < \cdots < \lambda_n$ we have $A_i + A_{i+1} + \cdots + A_n \subset \{g \geqslant \lambda_i\}$, and g being α-measurable: $\{g \geqslant \lambda_i\} \in \alpha$. herefore ${}^*C_i \subset \{g \geqslant \lambda_i\}$ a.e. and a fortiori $g \geqslant \lambda_i$ a.e. on ${}^*C_i - {}^*C_{i+1}$.

We now prove $L = L^*$. First observe that if L is the α-conditional ei of X, then $-L$ is the α^c-conditional es of $-X$. Then the relations

$$\{\text{essup}[C \in \alpha;\, C \subset A_k + \cdots + A_n]\}^c = \text{esinf}\{D \in \alpha^c;\, A_1 + \cdots + A_{k-1} \subset D\},$$

$U = U^*$ imply the result.

The last proposition and Theorem 3.4 imply the following corollary.

COROLLARY 4.2. *Assume the hypotheses of the preceding proposition and define* $M_{ij} = ({}^*C_i - {}^*C_{i+1}) \cap ({}_*C_j - {}_*C_{j+1})$, $i \geqslant j$. *Then* $d = \frac{1}{2}\sup\{\lambda_i - \lambda_j, P(M_{ij}) > 0\}$.

The following lemma notably simplifies the proof of Theorem 4.4, based on the technique used by Landers and Rogge in [12].

LEMMA 4.3. *For each α-measurable function g and every r, $1 \leqslant r < \infty$, we have*:

$$\|X - \min\{f_r, g\}\|_r \leqslant \|X - g\|_r.$$

Proof. Since $|X - \min\{f_r, g\}|^r + |X - \max\{f_r, g\}|^r = |X - f_r|^r + |X - g|^r$, by integration we obtain:

$$\int |X - \min\{f_r, g\}|^r \, dP + \int |X - \max\{f_r, g\}|^r \, dP$$

$$= \int |X - f_r|^r \, dP + \int |X - g|^r \, dP. \qquad (*)$$

As g is α-measurable, and hence $\min\{f_r, g\}$ and $\max\{f_r, g\}$ are α-measurable, we have by definition of f_r:

$$\int |X - f_r|^r \, dP$$

$$\leqslant \min\left\{\int |X - \min\{f_r, g\}|^r \, dP, \int |X - \max\{f_r, g\}|^r \, dP\right\},$$

hence $(*)$ yields $\int |X - \min\{f_r, g\}|^r \, dP \leqslant \int |X - g|^r \, dP$.

THEOREM 4.4. *For every sequence* $\{r(n)\}_n$, $r(n)\to\infty$, $\liminf f_{r(n)}$ *and* $\limsup f_{r(n)}$ *are variables in* $\mathscr{A}_\infty(X, B)$.

Proof. Suppose w.l.o.g. that $r(n)\leqslant r(n+1)$ for every n, and consider a fixed $r\geqslant 1$. Then there exists $n_0(=n_0(r))$ such that $r\leqslant r(n)$ for all $n\geqslant n_0$; hence, if $k\geqslant n_0$, the repeated use of the previous lemma gives:

$$\begin{aligned}
&\|X-\min\{f_{r(1)}, k\leqslant l\leqslant n\}\|_r\\
&\leqslant \|X-\min\{f_{r(1)}, k\leqslant l\leqslant n\}\|_{r(k)}\\
&\leqslant \|X-\min\{f_{r(1)}, k+1\leqslant l\leqslant n\}\|_{r(k)}\\
&\leqslant \|X-\min\{f_{r(1)}, k+1\leqslant l\leqslant n\}\|_{r(k+1)}\\
&\leqslant \cdots \leqslant \|X-f_{r(n)}\|_{r(n)}\leqslant \|X-f_\infty\|_{r(n)}\\
&\leqslant \|X-f_\infty\|_\infty
\end{aligned}$$

for any f_∞ in $\mathscr{A}_\infty(X, B)$ (not empty from Theorem 3.3).

Hence $n\to\infty$ yields $\|X-\inf\{f_{r(1)}, k\leqslant l\}\|_r\leqslant\|X-f_\infty\|_\infty$, whence $k\to\infty$ yields $\|X-\liminf f_{r(n)}\|_r\leqslant\|X-f_\infty\|_\infty$. Now $r\to\infty$ yields that Theorem 4.4 (the assertion $\limsup f_{r(n)}\in\mathscr{A}_\infty(X, B)$ can be proved in the same way from the obvious modification of Lemma 4.3).

It may be suspected that Theorem 4.4 may be improved in some sense. For example, in the case in which α is a σ-algebra, the α-conditional midrange plays an important role as the best of the best L_∞-approximations. Is it true that, in isotonic approximation, the α-conditional midrange, M_α, verifies $\liminf f_{r(n)}\leqslant M_\alpha\leqslant\limsup f_{r(n)}$ a.e. for every sequence $r(n)\to\infty$? The answer is negative. It is even possible that $f_{r(n)}$ converges for every sequence $r(n)\to\infty$ and $P\{M_\alpha\neq\lim f_{r(n)}\}>0$:

Consider the probability space $([0,1],\beta,\ell)$, where ℓ is the Lebesgue measure on β (the Borel sets in $[0,1]$).

Let X be the random variable $X=I_{[0,10^{-2}]}-I_{(10^{-2},10^{-1}]}$ and let $\alpha=\{(a,1],[a,1];\ a\in[0,1]\}$. It is well known that the increasing Borel functions on $[0,1]$ are the α-measurable functions on this space. Let $r\in(1,\infty)$; it is easy to prove that $f_r(\omega)\leqslant 0$ if $\omega\in[0,10^{-1}]$ and $f_r(\omega)=0$ if $\omega\in(10^{-1},1]$. Moreover, if $r(n)\to\infty$ then $\lim f_{r(n)}=0$ a.e. On the other hand, the α-conditional midrange of X is $M_\alpha=\frac{1}{2}I_{(1/10,1]}$.

Finally we prove the convergence of the L_r-distance from X to $L_r(\alpha)$, $d_r=\|X-f_r\|_r$, to $d=\|X-f_\infty\|_\infty$, as $r\to\infty$.

THEOREM 4.5. *Let* d_r *be the* L_r*-distance from* X *to* $L_r(\alpha)$. *Then* $d_r\uparrow d$ *as* $r\uparrow\infty$.

Proof. It suffices to prove $\lim_{n\to\infty} d_{r(n)}\geqslant d$. Let $r(0)$ be arbitrary but fixed. It suffices to show, according to Theorem 4.4, that $\|X - \liminf_n f_{r(n)}\|_{r(0)} \leqslant \lim_{n\to\infty} d_{r(n)}$.

Assume w.l.o.g. that $r(n) \geqslant r(0)$. Let $m \geqslant k$, then (see Theorem 4.4)

$$\begin{aligned} &\|X - \min\{f_{r(1)}, k \leqslant l \leqslant m\}\|_{r(0)} \\ &\quad \leqslant \|X - \min\{f_{r(1)}, k \leqslant l \leqslant m\}\|_{r(k)} \\ &\quad \leqslant \|X - f_{r(m)}\|_{r(m)} \leqslant \lim_{n\to\infty} d_{r(n)}. \end{aligned}$$

Hence $\|X - \min\{f_{r(1)}, k \leqslant l\}\|_{r(0)} \leqslant \lim_{n\to\infty} d_{r(n)}$, whence

$$\|X - \liminf_n f_{r(n)}\|_{r(0)} \leqslant \lim_{n\to\infty} d_{r(n)}.$$

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